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The linear convergence of limit periodic continued fractions

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Abstract: The only linearly convergent continued fractions are the limit periodic ones.

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Let us consider the continued fraction

$$\frac{a_1|}{|1} + \frac{a_2|}{|1} + \cdots$$

where the a_n are complex numbers.

Let $C_n = A_n/B_n$ be its convergents. We set $h_n = B_n/B_{n-1}$. We assume that $\lim_{n\to\infty} C_n = C$ is finite.

Theorem 1. *If* $\exists a \in \mathbb{C}$, $a \neq -\frac{1}{4} + c$ *with* $c \leqslant 0$ *such that* $\lim_{n\to\infty} a_n = a$ *then* $\exists r \in \mathbb{C}$, $|r| < 1$ *such that* $\lim_{n\to\infty}(C_n - C)/(C_{n-1} - C) = r$.

Proof. If $a \neq -\frac{1}{4} + c$ with $c \leqslant 0$ the two zeros of $x^2 - x - a = 0$ have distinct moduli. Since $B_{n+1} = B_n + a_{n+1}B_{n-1}$ then, by Poincaré's theorem [9], $\exists h \in \mathbb{C}$ such that $\lim_{n\to\infty} h_n = h$. Moreover it is easy to see that $|h| \neq (\frac{1}{4} - c)^{1/2}$. But we have [10]

$$\Delta C_n/\Delta C_{n-1} = -1 + 1/h_{n+1}. \tag{1}$$

Thus $\exists r \in \mathbb{C}$ such that $\lim_{n\to\infty} \Delta C_n/\Delta C_{n-1} = r = -1 + 1/h$. Moreover, since h is equal to the zero of greatest modulus of $x^2 - x - a = 0$, then $|r| < 1$. Then, by a result of Delahaye [2], $\lim_{n\to\infty}(C_n - C)/(C_{n-1} - C) = r$. □

This theorem was given in [7] with a different proof. See also [8].

Let us now give the reciprocal of this result:

Theorem 2. *If* $\exists r \in \mathbb{C}$, $r \neq -1$ *such that* $\lim_{n\to\infty} \Delta C_n/\Delta C_{n-1} = r$ *then* $\exists a \in \mathbb{C}$, *such that* $\lim_{n\to\infty} a_n = a = -r/(1+r)^2$.

Proof. From (1) we see that, if $r \neq -1$, $\exists h \neq 0$ and finite such that $\lim_{n\to\infty} h_n = h$. But $h_{n+1} = 1 + a_{n+1}/h_n$ or

$$h_n(h_{n+1} - 1) = a_{n+1}$$

which shows that $\exists a \in \mathbb{C}$ such that $\lim_{n\to\infty} a_n = a$. □

Let us study this reciprocal in more detail. As we saw before r and h are related by $h = (1+r)^{-1}$. If $r = e^{i\theta}$, that is if $|r| = 1$,

$$h = \tfrac{1}{2} - i \sin\theta/2(1+\cos\theta).$$

Hence $|h|^2 = \frac{1}{4} + \sin^2\theta/4(1+\cos\theta)^2 = \frac{1}{4} - c$ with $c \leqslant 0$. Thus, if $|r| \neq 1$, then $|h| \neq (\frac{1}{4} - c)^{1/2}$ with $c \leqslant 0$. Let us now examine $|a|$. If $|r| = 1$, then

$$a = -\left(\tfrac{1}{2} - i\frac{\sin\theta}{2(1+\cos\theta)}\right)\left(\tfrac{1}{2} + i\frac{\sin\theta}{2(1+\cos\theta)}\right)$$

$$= -\tfrac{1}{4} - \frac{\sin^2\theta}{4(1+\cos\theta)^2} = -\tfrac{1}{4} + c.$$

Finally, if $|r| \neq 1$ then $a \neq -\frac{1}{4} + c$ with $c \leqslant 0$. This last result can be gathered with that of Theorem 1 and we get the:

Theorem 3. *A necessary and sufficient condition that* $\exists r \in \mathbb{C}$, $|r| < 1$ *such that* $\lim_{n\to\infty}(C_n - C)/(C_{n-1} - C) = r$ *is that* $\exists a \in \mathbb{C}$, $a \neq -\frac{1}{4} + c$ *with* $c \leq 0$ *such that* $\lim_{n\to\infty} a_n = a$. *Moreover* a *and* r *are related by* $a = -r/(1+r)^2$.

Proof. If $\exists C \in \mathbb{C}$, $\exists r \in \mathbb{C}$, $r \neq 1$ such that $\lim_{n\to\infty} C_n = C$ and $\lim_{n\to\infty}(C_n - C)/(C_{n-1} - C) = r$ then, by a result due to Delahaye [2], the ratio $\Delta C_n/\Delta C_{n-1}$ has a limit and this limit is equal to r. By Theorem 2, if $r \neq -1$, the continued fraction is limit periodic. Moreover, as we saw above, if $|r| \neq 1$ then $a \neq -\frac{1}{4} + c$ with $c \leqslant 0$ and the first part of the result follows from Theorem 1. The reciprocal is Theorem 1. □

Remarks. Let us make some remarks on the respective values of a and r:

(i) $r = 0$ if and only if $a = 0$. Since $r = -1 + 1/h$, r is zero if and only if h equals 1. If $h = 1$ then $h(h-1) = a = 0$. Reciprocally if $a = 0$, the zeros of $x^2 - x - a = 0$ are 0 and 1 and thus, by Poincaré's theorem, h is 0 or 1. If $h = 0$ then r is infinite which is impossible. Thus $h = 1$ which gives $r = 0$. Thus limit periodic continued fractions converge super linearly if and only if $\lim_{n\to\infty} a_n = 0$. In that case it is less crucial to be able to accelerate the convergence.

(ii) If $r = 1$ then $a = -\frac{1}{4}$. This is the worth case since the convergence, when it occurs, is very slow (logarithmic convergence). Reciprocally if $a = -\frac{1}{4}$, the two zeros of $x^2 - x + \frac{1}{4} = 0$ are equal to $\frac{1}{2}$ and Poincaré's theorem does not allow to conclude.

(iii) Another proof of Theorem 3 by using properties of linear functional transformations and the Koebe function was given to us by Waadeland.

Theorem 3 has important consequences concerning the convergence acceleration of limit periodic continued fractions. Since such fractions are linearly converging if $a \neq 0$ (if $a = 0$, the continued fraction converges super linearly and, thus, is less important to accelerate) they can be accelerated in many different ways such as modifications, see [5] for a review, or various sequence transformations, [1,6].

On the other hand, continued fractions which are not 1-limit periodic will be difficult to accelerate. This follows from the theory of remanence of a set of sequences introduced by Delahaye and Germain Bonne [4]. It means that a *universal* algorithmic method for transforming (C_n) into another sequence converging faster cannot exist for all continued fractions which are not 1-limit periodic (by algorithmic method it is meant a method which does not depend on asymptotic properties of (C_n) but only on a finite number of its terms). Subsets of such continued fractions will have to be considered. Even in the case where the ratios $(C_n - C)/(C_{n-1} - C)$ remain bounded from below and above such a universal transformation cannot exist [3].

Finally let us mention that some similar results seem to exist for limit k-periodic continued fractions. For example it is easy to see that the even and odd parts of a limit 2-periodic continued fraction are limit periodic with the same asymptotic error coefficient. Obviously, by our Theorem 3, the converse is false.

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